

The covariant graviton propagator in de Sitter spacetime

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Abstract

We consider the covariant graviton propagator in de Sitter spacetime in a gauge with two parameters, α and β , in the Euclidean approach. We give an explicit form of the propagator with a particular choice of β but with arbitrary value of α . We confirm that two-point functions of local gauge-invariant quantities do not increase as the separation of the two points becomes large.

1 Introduction

Quantum field theory in de Sitter spacetime (see, e.g., [1]) has been studied extensively because of its relevance to inflationary cosmologies [2, 3, 4]. The graviton two-point functions, which represent correlation of vacuum fluctuation in the gravitational field, have been studied by many authors using various gauges in this spacetime (see, e.g., [5]–[10]). These two-point functions are known to increase as the (coordinate) distance between the two points becomes large. Some authors have suggested that this behaviour may manifest itself in the two-point functions of gauge invariant quantities.

However, a non-covariant two-point function which does not increase as a function of two-point distance was obtained recently in open de Sitter spacetime [11]. Subsequently,

the present authors showed that the logarithmically increasing term of the non-covariant graviton two-point function in the spatially-flat coordinates found by Allen [12] is a gauge artifact [13]. It is desirable to extend these observations to covariant gauges, which are more suitable for computation. The present authors have started investigation in this direction and shown that the pure-trace part of a covariant two-point function, whose growth with distance was suggested to be physical [6, 8], gives a pure-gauge contribution when combined with another part [14, 15].

In this paper we go beyond that work and consider the full two-point function in a covariant gauge with two parameters. (This gauge was previously used in [10], but the explicit form of the corresponding two-point function has not been written down.)

We work in the Euclidean approach of Allen and Turyn [6]. (Thus, what we compute is the Green function on the 4-sphere, which becomes the Feynman propagator in de Sitter spacetime upon analytic continuation. It can be identified with the Wightman two-point function for two points which are not causally connected.) These authors chose the gauge in which the flat-space limit is simplest and found that the propagator increases as the separation between the two points becomes large. We generalize their work and find the propagator which depends on two gauge parameters, α and β . Then we write down the propagator explicitly for a particular choice of β with the value of α left arbitrary. (Our propagator consists of three sectors: the transverse-traceless, vector and scalar sectors. The transverse-traceless and vector sectors are the same as those in [6]. We need to generalize only the scalar sector.) Unfortunately, we cannot find any choice of gauge parameters that eliminates the large-distance growth of the propagator. Nevertheless, we show that this growth will not be reflected in the two-point function of a local gauge-invariant quantity, as expected from the results in non-covariant gauges mentioned before.

The rest of the paper is organized as follows. In section 2 we discuss the general structure of the Green function following Allen and Turyn. We write down their results for the transverse-traceless and vector sectors of the propagator in section 3 for completeness, and present our result for the scalar sector in section 4. In the final section we summarize this paper and show that the two-point function of a local gauge-invariant quantity is bounded as the separation of the two points becomes large. We use natural units $\hbar = c = 1$ throughout this paper.

2 The field equation and the Green function

De Sitter spacetime is a contracting and expanding 3-sphere with the following line element:

$$ds^2 = -dt^2 + \frac{1}{H^2} \cosh^2(Ht) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (1)$$

where H is the Hubble constant. By introducing the variable $\tau \equiv \pi/2 - iHt$, we obtain

$$ds^2 = H^{-2} \left\{ d\tau^2 + \sin^2 \tau [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \right\}, \quad (2)$$

which is the line element of a four-dimensional sphere (S^4) of radius H^{-1} . We will compute the Green function of the linearized gravity on this space, which becomes the Feynman propagator in de Sitter spacetime upon analytic continuation. (For this reason we will use the terms “Green function” and “propagator” interchangeably.) Then, we are automatically choosing the Euclidean [16], or Bunch-Davies [17], vacuum.

We start from the Lagrangian density of pure gravity with positive cosmological constant,

$$\mathcal{L}_{\text{full}} = \sqrt{-\tilde{g}} (R - 6H^2), \quad (3)$$

where \tilde{g}_{ab} is the full metric and where R is the corresponding scalar curvature. We write the metric as $\tilde{g}_{ab} = g_{ab} + h_{ab}$, where g_{ab} is the background de Sitter metric given by (1), and expand the Lagrangian (3) to second order in h_{ab} . The Lagrangian density for the linearized gravity thus obtained is written, after dropping a total divergence, as

$$\begin{aligned} \mathcal{L}_{\text{inv}} = & \sqrt{-g} \left[\frac{1}{2} \nabla_a h^{ac} \nabla^b h_{bc} - \frac{1}{4} \nabla_a h_{bc} \nabla^a h^{bc} + \frac{1}{4} (\nabla^a h - 2 \nabla^b h^a_b) \nabla_a h \right. \\ & \left. - \frac{1}{2} H^2 \left(h_{ab} h^{ab} + \frac{1}{2} h^2 \right) \right] \end{aligned} \quad (4)$$

with $h = h^a_a$. Here the indices are raised and lowered by g_{ab} , and the ∇_a denotes the background covariant derivative. This Lagrangian density is invariant under the gauge transformation

$$h_{ab} \rightarrow h_{ab} + \nabla_a \Lambda_b + \nabla_b \Lambda_a$$

up to a total divergence. One needs to break this gauge invariance for canonical quantization. For this purpose we add the following gauge-fixing term in the Lagrangian density:

$$\mathcal{L}_{\text{gf}} = -\frac{\sqrt{-g}}{2\alpha} \left(\nabla_a h^{ab} - \frac{1+\beta}{\beta} \nabla^b h \right) \left(\nabla^c h_{cb} - \frac{1+\beta}{\beta} \nabla_b h \right). \quad (5)$$

Then the Euler-Lagrange field equations derived from $\mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gf}}$ are

$$L_{ab}{}^{cd} h_{cd} \equiv \frac{1}{2} \square h_{ab} - \left(\frac{1}{2} - \frac{1}{2\alpha} \right) (\nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a)$$

$$\begin{aligned}
& + \left[\frac{1}{2} - \frac{\beta+1}{\alpha\beta} \right] \nabla_a \nabla_b h + \left[\frac{(\beta+1)^2}{\alpha\beta^2} - \frac{1}{2} \right] g_{ab} \square h \\
& + \frac{1}{2} g_{ab} \left(1 - \frac{2(1+\beta)}{\alpha\beta} \right) \nabla_c \nabla_d h^{cd} - H^2 \left(h_{ab} + \frac{1}{2} g_{ab} h \right) = 0. \quad (6)
\end{aligned}$$

To find the Green function of the operator $-L_{ab}{}^{cd}$ on S^4 we need the symmetric tensor eigenfunctions of the Laplace-Beltrami operator $\square = \nabla_a \nabla^a$ described in [6]. It is convenient to define the inner product of any two symmetric tensors $h_{ab}^{(1)}$ and $h_{ab}^{(2)}$ on S^4 as

$$(h^{(1)}, h^{(2)})_T = \int_{S^4} dx \, \overline{h_{ab}^{(1)}} h^{(2)ab}, \quad (7)$$

where dx is the volume element. The inner product of scalar functions, $(\cdot, \cdot)_S$, and that of vector functions, $(\cdot, \cdot)_V$, are defined in a similar manner.

The scalar eigenfunctions $\phi^{(n,i)}$ satisfy

$$\square \phi^{(n,i)} = -\lambda_n \phi^{(n,i)}, \quad (8)$$

where $\lambda_n = n(n+3)H^2$, $n = 0, 1, 2, \dots$, and where the index i distinguishes the modes with the same eigenvalue λ_n . (It is well known that the spectrum of the operator \square on the N -sphere of radius H^{-1} is given by $-n(n+N-1)H^2$, where n is a non-negative integer. See, e.g., [18].) We impose the normalization condition $(\phi^{(n,i)}, \phi^{(m,j)})_S = \delta^{nm} \delta^{ij}$. The pure-trace eigenfunctions of the operator \square on the symmetric tensors are

$$\chi_{ab}^{(n,i)} = \frac{1}{2} g_{ab} \phi^{(n,i)}. \quad (9)$$

The traceless eigenfunctions that can be expressed in terms of scalars are

$$W_{ab}^{(n,i)} = \frac{2(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square) \phi^{(n,i)}}{\sqrt{3\lambda_n(\lambda_n - 4H^2)}}, \quad (n \geq 2). \quad (10)$$

The modes with $n \leq 1$ are missing because the differential operator $\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square$ annihilates $\phi^{(n,i)}$ with $n \leq 1$. These tensor eigenfunctions satisfy $(\chi^{(n,i)}, W^{(m,j)})_T = 0$ and $(\chi^{(n,i)}, \chi^{(m,j)})_T = (W^{(n,i)}, W^{(m,j)})_T = \delta^{nm} \delta^{ij}$. The divergence-free vector eigenfunctions $\xi^{(n,i)}$ satisfy [18]

$$\square \xi^{(n,i)} = -(\lambda_n - H^2) \xi^{(n,i)}, \quad (n \geq 1). \quad (11)$$

We impose $(\xi^{(n,i)}, \xi^{(m,j)})_V = \delta^{nm} \delta^{ij}$. The symmetric tensor eigenfunctions that are expressed in terms of these vectors are

$$V_{ab}^{(n,i)} = \frac{\nabla_a \xi_b^{(n,i)} + \nabla_b \xi_a^{(n,i)}}{\sqrt{2(\lambda_n - 4H^2)}}, \quad (n \geq 2). \quad (12)$$

The modes with $n = 1$ are missing because the vectors $\xi_a^{(1,i)}$ satisfy $\nabla_a \xi_b^{(1,i)} + \nabla_b \xi_a^{(1,i)} = 0$. These tensor eigenfunctions are orthogonal to the modes discussed before with respect to the inner product $(\cdot, \cdot)_T$ and satisfy $(V^{(n,i)}, V^{(m,j)})_T = \delta^{nm} \delta^{ij}$. The traceless and divergence-free (transverse-traceless) modes $h_{ab}^{(TT,n,i)}$ satisfy [18]

$$\square h_{ab}^{(TT,i,n)} = -(\lambda_n - 2H^2) h_{ab}^{(TT,i,n)}, \quad (n \geq 2), \quad (13)$$

and $(h^{(TT,n,i)}, h^{(TT,m,j)})_T = \delta^{nm} \delta^{ij}$. These modes are orthogonal to the modes $\chi_{ab}^{(n,i)}$, $W_{ab}^{(n,i)}$ and $V_{ab}^{(n,i)}$ with respect to $(\cdot, \cdot)_T$. Any symmetric tensor field on S^4 can be expanded in terms of these eigenfunctions.

Now, define the δ -function $\delta_{aba'b'}(x, x')$ by

$$\int \delta_{aba'b'}(x, x') f^{a'b'}(x') dx' = f_{ab}(x) \quad (14)$$

for any smooth symmetric tensor f_{ab} on S^4 . Here, dx' is the volume element. Then, using the completeness of the modes described before, we can express the δ -function as

$$\delta_{aba'b'}(x, x') = \delta_{aba'b'}^{(TT)}(x, x') + \delta_{aba'b'}^{(V)}(x, x') + \delta_{aba'b'}^{(S)}(x, x'), \quad (15)$$

where

$$\delta_{aba'b'}^{(TT)}(x, x') = \sum_{n=2}^{\infty} \sum_i h_{ab}^{(TT,n,i)} \overline{h_{a'b'}^{(TT,n,i)}(x')}, \quad (16)$$

$$\delta_{aba'b'}^{(V)}(x, x') = \sum_{n=2}^{\infty} \sum_i V_{ab}^{(n,i)}(x) \overline{V_{a'b'}^{(n,i)}(x')}, \quad (17)$$

$$\delta_{aba'b'}^{(S)}(x, x') = \sum_{n=2}^{\infty} \sum_i W_{ab}^{(n,i)}(x) \overline{W_{a'b'}^{(n,i)}(x')} + \sum_{n=0}^{\infty} \sum_i \chi_{ab}^{(n,i)} \overline{\chi_{a'b'}^{(n,i)}(x')}. \quad (18)$$

The Green function $G_{aba'b'}(x, x')$ of the operator $-L_{ab}{}^{cd}$ given by (6) is defined by the equation

$$-L_x{}^{abcd} G_{cda'b'}(x, x') = \delta_{a'b'}^{ab}(x, x'). \quad (19)$$

Here, the subscript x in $L_x{}^{abcd}$ indicates that the differential operator L^{abcd} acts at x rather than at x' .

It can readily be seen that the Green function can be written in the form

$$G_{aba'b'}(x, x') = G_{aba'b'}^{(TT)}(x, x') + G_{aba'b'}^{(V)}(x, x') + G_{aba'b'}^{(S)}(x, x'), \quad (20)$$

with

$$G_{aba'b'}^{(TT)}(x, x') = \sum_{n=2}^{\infty} \sum_i a_n h_{ab}^{(TT,n,i)} \overline{h_{a'b'}^{(TT,n,i)}(x')}, \quad (21)$$

$$G_{aba'b'}^{(V)}(x, x') = \sum_{n=2}^{\infty} \sum_i b_n V_{ab}^{(n,i)}(x) \overline{V_{a'b'}^{(n,i)}(x')}, \quad (22)$$

$$\begin{aligned} G_{aba'b'}^{(S)}(x, x') &= \sum_{n=0}^{\infty} \sum_i c_n \chi_{ab}^{(n,i)} \overline{\chi_{a'b'}^{(n,i)}(x')} + \sum_{n=2}^{\infty} \sum_i e_n W_{ab}^{(n,i)}(x) \overline{W_{a'b'}^{(n,i)}(x')} \\ &\quad + \sum_{n=2}^{\infty} \sum_i d_n \left[W_{ab}^{(n,i)}(x) \overline{\chi_{a'b'}^{(n,i)}(x')} + \chi_{ab}^{(n,i)}(x) \overline{W_{a'b'}^{(n,i)}(x')} \right], \end{aligned} \quad (23)$$

where a_n, b_n, c_n, d_n and e_n are constants. We call the bi-tensors $G_{aba'b'}^{(TT)}$ and $G_{aba'b'}^{(V)}$ the transverse-traceless and vector sectors, respectively, and the $G_{aba'b'}^{(S)}$ the scalar sector of the propagator. It is clear that equation (19) is satisfied if

$$-L_{ab}{}^{cd} G_{cda'b'}^{(TT)}(x, x') = \delta_{aba'b'}^{(TT)}(x, x'), \quad (24)$$

$$-L_{ab}{}^{cd} G_{cda'b'}^{(V)}(x, x') = \delta_{aba'b'}^{(V)}(x, x'), \quad (25)$$

$$-L_{ab}{}^{cd} G_{cda'b'}^{(S)}(x, x') = \delta_{aba'b'}^{(S)}(x, x'). \quad (26)$$

We discuss the solutions of these equations in the next two sections.

3 The transverse-traceless and vector sectors

By applying the operator (6) on the modes $h_{ab}^{(TT,n,i)}$ and $V_{ab}^{(n,i)}$ in the transverse-traceless and vector sectors we find

$$-L_{ab}{}^{cd} h_{cd}^{(TT,n,i)} = \frac{\lambda_n}{2} h_{ab}^{(TT,n,i)}, \quad (27)$$

$$-L_{ab}{}^{cd} V_{cd}^{(n,i)} = \frac{1}{2\alpha} (\lambda_n - 4H^2) V_{ab}^{(n,i)}. \quad (28)$$

One finds equation (28) easily by noting that the part of equation (6) coming from the Lagrangian density \mathcal{L}_{inv} vanishes because $V_{ab}^{(n,i)} \propto \nabla_a \xi_b^{(n,i)} + \nabla_b \xi_a^{(n,i)}$. By using (27) and (28) in (24) and (25) we find $a_n = 2\lambda_n^{-1}$ and $b_n = 2\alpha[\lambda_n - 4H^2]^{-1}$. Hence,

$$G_{aba'b'}^{(TT)}(x, x') = 2 \sum_{n=2}^{\infty} \sum_j \frac{h_{ab}^{(TT,n,j)}(x) \overline{h_{a'b'}^{(TT,n,j)}(x')}}{\lambda_n}, \quad (29)$$

$$G_{aba'b'}^{(V)}(x, x') = 2\alpha \sum_{n=2}^{\infty} \sum_j \frac{V_{ab}^{(n,j)}(x) \overline{V_{a'b'}^{(n,j)}(x')}}{\lambda_n - 4H^2}. \quad (30)$$

Closed-form expressions for $G_{aba'b'}^{(TT)}(x, x')$ and $G_{aba'b'}^{(V)}(x, x')$ (with $\alpha = 1$) have been derived by Allen and Turyn [6]. We need to quote some definitions given in [19] to state their results. We define $\mu(x, x')$ to be the geodesic distance on S^4 between the

two points x and x' . Then, the vectors $n_a \equiv \nabla_a \mu$, where the derivative operator acts at x , and $n_{a'} \equiv \nabla_{a'} \mu$, where it acts at x' , are the tangent vectors to the geodesic. (Primed indices refer to the tangent space at x' and unprimed indices refer to that at x here and in the rest of this paper.) Primed indices are raised and lowered by the metric $g_{a'b'}$ at x' and unprimed ones by the metric g_{ab} at x . The parallel propagator $g^a_{a'}$ is defined as follows: given a vector $Y^{a'}$ at x' , the vector $g^a_{a'} Y^{a'}$ is obtained by parallelly transporting $Y^{a'}$ along the geodesic to the point x . We also define the variable $z = \cos^2(H\mu/2)$. Large spacelike separation in de Sitter spacetime corresponds to the limit $z \rightarrow -\infty$ [19]. (Although there is no spacelike geodesic connecting the two points for $z < -1$, large spacelike *coordinate* distance in the metric $ds^2 = (H\lambda)^{-2}(-d\lambda^2 + d\mathbf{x}^2)$ corresponds to this limit.) We define further the following traceless bi-tensors:

$$T_{aba'b'}^{(1)} = \left(n_a n_b - \frac{1}{4} g_{ab} \right) \left(n_{a'} n_{b'} - \frac{1}{4} g_{a'b'} \right), \quad (31)$$

$$T_{aba'b'}^{(2)} = g_{aa'} g_{bb'} + g_{a'b} g_{b'a} - \frac{1}{2} g_{ab} g_{a'b'}, \quad (32)$$

$$T_{aba'b'}^{(3)} = g_{aa'} n_b n_{b'} + g_{ab'} n_b n_{a'} + g_{ba'} n_b n_{b'} + g_{bb'} n_a n_{a'} + 4 n_a n_b n_{a'} n_{b'}. \quad (33)$$

With these definitions the bi-tensor $G_{aba'b'}^{(TV)} \equiv G_{aba'b'}^{(TT)}(x, x') + G_{aba'b'}^{(V)}(x, x')$ can be written as

$$G_{aba'b'}^{(TV)}(x, x') = \frac{H^2}{16\pi^2} \left[f^{(TV,1)}(z) T_{aba'b'}^{(1)} + f^{(TV,2)}(z) T_{aba'b'}^{(2)} + f^{(TV,3)}(z) T_{aba'b'}^{(3)} \right], \quad (34)$$

where

$$f^{(TV,1)}(z) = -\frac{16}{9} + \frac{8\alpha}{3} + \frac{4-4\alpha}{1-z} + \left(1 - \frac{3\alpha}{5} \right) \left[-\frac{8}{3} \left(2 - \frac{1}{z^2} - \frac{1}{z^3} \right) \log(1-z) + \frac{4}{z} + \frac{8}{3z^2} \right], \quad (35)$$

$$f^{(TV,2)}(z) = \frac{2}{3} - \frac{13\alpha}{5} + \frac{1+3\alpha}{6(1-z)} + \left(1 - \frac{3\alpha}{5} \right) \left[\left(2 + \frac{1}{9z^2} + \frac{1}{9z^3} \right) \log(1-z) + \frac{1}{6z} + \frac{1}{9z^2} \right], \quad (36)$$

$$f^{(TV,3)}(z) = \frac{6\alpha z}{5} + \frac{2}{3} - \frac{16\alpha}{5} - \frac{2}{3(1-z)} + \left(1 - \frac{3\alpha}{5} \right) \left[\left(2 - \frac{10}{9z} - \frac{4}{9z^2} - \frac{4}{9z^3} \right) \log(1-z) - \frac{2}{3z} - \frac{4}{9z^2} \right]. \quad (37)$$

(The $G_{aba'b'}^{(TV)}$ here is twice as large as that of Allen and Turyn due to the difference in the normalization of the field h_{ab} .) It is interesting to note that logarithmic terms are absent in $G_{aba'b'}^{(TV)}(x, x')$ if we choose $\alpha = 5/3$. Notice, however, that there is a term linear in z unless $\alpha = 0$.

4 The scalar sector

Application of the operator defined by (6) on the tensor eigenfunctions appearing in the scalar sector yields, after a long but straightforward calculation,

$$-L_{ab}{}^{cd}\chi_{cd}^{(n,i)} = K_{11}^{(n)}\chi_{ab}^{(n,i)} + K_{12}^{(n)}W_{ab}^{(n,i)}, \quad (38)$$

$$-L_{ab}{}^{cd}W_{cd}^{(n,i)} = K_{12}^{(n)}\chi_{ab}^{(n,i)} + K_{22}^{(n)}W_{ab}^{(n,i)}, \quad (39)$$

where

$$K_{11}^{(n)} = \left[\frac{1}{\alpha} \left(\frac{3}{2} + \frac{2}{\beta} \right)^2 - \frac{3}{4} \right] \lambda_n + 3H^2, \quad (40)$$

$$K_{12}^{(n)} = - \left(\frac{\alpha - 3}{4\alpha} - \frac{1}{\alpha\beta} \right) \sqrt{3\lambda_n(\lambda_n - 4H^2)}, \quad (41)$$

$$K_{22}^{(n)} = - \frac{\alpha - 3}{4\alpha} \left(\lambda_n + \frac{12H^2}{\alpha - 3} \right). \quad (42)$$

The equation $-L_{ab}{}^{cd}G_{cda'b'}^{(S)}(x, x') = \delta_{aba'b'}^{(S)}(x, x')$ is satisfied if

$$\begin{pmatrix} c_n & d_n \\ d_n & e_n \end{pmatrix} = \begin{pmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ K_{12}^{(n)} & K_{22}^{(n)} \end{pmatrix}^{-1}. \quad (43)$$

One can readily find the $G_{aba'b'}^{(S)}(x, x')$ by substituting the coefficients c_n , d_n and e_n thus obtained¹ in (23) and recalling the definitions of $\chi_{ab}^{(n,i)}$ and $W_{ab}^{(n,i)}$ given by (9) and (10). Let us define

$$\Delta_{m^2}(x, x') = \sum_{n=0}^{\infty} \sum_i \frac{\phi^{(n,i)}(x) \overline{\phi^{(n,i)}(x')}}{\lambda_n + m^2 H^2}, \quad (44)$$

$$\Delta_{m^2}^-(x, x') = \sum_{n=1}^{\infty} \sum_i \frac{\phi^{(n,i)}(x) \overline{\phi^{(n,i)}(x')}}{\lambda_n + m^2 H^2}, \quad (45)$$

$$\Delta_{m^2}^{--}(x, x') = \sum_{n=2}^{\infty} \sum_i \frac{\phi^{(n,i)}(x) \overline{\phi^{(n,i)}(x')}}{\lambda_n + m^2 H^2}, \quad (46)$$

$$\Delta_{m^2}^{(1)}(x, x') = \sum_{n=0}^{\infty} \sum_i \frac{\phi^{(n,i)}(x) \overline{\phi^{(n,i)}(x')}}{(\lambda_n + m^2 H^2)^2}. \quad (47)$$

Then, we can write

$$G_{aba'b'}^{(S)}(x, x') = \left(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) \left(\nabla_{a'} \nabla_{b'} - \frac{1}{4} g_{a'b'} \square' \right) A_n^{(1)}(x, x')$$

¹The cases with $n = 0$ and 1 need to be treated separately, but the expressions for c_n for these cases turn out to be the same as the other cases.

$$\begin{aligned}
& + \frac{1}{2} \left[\left(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) g_{a'b'} + g_{ab} \left(\nabla_{a'} \nabla_{b'} - \frac{1}{4} g_{a'b'} \square' \right) \right] A^{(2)}(x, x') \\
& + \frac{1}{4} g_{ab} g_{a'b'} A_n^{(3)}(x, x'), \tag{48}
\end{aligned}$$

where

$$A^{(1)} = \frac{\alpha}{9H^4} \Delta_0^- - \frac{1}{3H^4} \Delta_{-4}^{--} + \frac{3-\alpha}{9H^4} \Delta_{3\beta} + \frac{4-(\alpha-3)\beta}{3H^2} \Delta_{3\beta}^{(1)}, \tag{49}$$

$$A^{(2)} = \frac{\beta[4-(\alpha-3)\beta]}{2} \Delta_{3\beta}^{(1)}, \tag{50}$$

$$A^{(3)} = \frac{\beta^2}{4} \left\{ (\alpha-3) \Delta_{3\beta} + 3[4-(\alpha-3)\beta] H^2 \Delta_{3\beta}^{(1)} \right\}. \tag{51}$$

(The operators \square and \square' act at x and x' , respectively.) We have used the fact that the differential operator $\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square$ annihilates the modes $\phi^{(n,i)}(x)$ with $n = 0$ or 1 .

One can explicitly evaluate $G_{aba'b'}^{(S)}(x, x')$ in terms of the variable $z = \cos^2(H\mu/2)$ by using [19]

$$\Delta_{m^2} = \frac{H^2}{16\pi^2} \Gamma(a_+) \Gamma(a_-) F(a_+, a_-; 2; z), \tag{52}$$

where

$$a_{\pm} = \frac{3}{2} \pm \left(\frac{9}{4} - m^2 \right)^{1/2}, \tag{53}$$

and [6]

$$\Delta_{m^2}^{(1)} = -H^{-2} \frac{\partial}{\partial c} \Delta_c(x, x')|_{c=m^2}, \tag{54}$$

$$\Delta_0^- = \frac{H^2}{16\pi^2} \left[\frac{1}{1-z} - 2 \log(1-z) - \frac{14}{3} \right], \tag{55}$$

$$\Delta_{-4}^{--} = \frac{H^2}{16\pi^2} \left[\frac{1}{1-z} + 6(1-2z) \log(1-z) + \frac{1}{10}(67-224z) \right]. \tag{56}$$

Here $\Gamma(x)$ is the gamma function and $F(a, b; c; x)$ is Gauss' hypergeometric function. The modulus of the function Δ_{m^2} behaves like $|z|^{-3/2}$ if $m^2 \geq 3/2$ and like $|z|^{-a_-}$ if $m^2 < 3/2$ for large $|z|$ provided that a_- is not zero or a negative integer. The gauge used by Allen and Turyn corresponds to the choice $\alpha = 1$, $\beta = -2$. Therefore their propagator involves the function Δ_{-6} which increases like $|z|^{(\sqrt{33}-3)/2}$ at large $|z|$. One can make the scalar sector $G_{aba'b'}^{(S)}$ decrease for large $|z|$ by choosing $\beta > 0$ [15]. The choice $\beta = 2/3$ is particularly convenient since the functions $\Delta_{3\beta}$ and $\Delta_{3\beta}^{(1)}$ in this case are

$$\Delta_2 = \frac{H^2}{16\pi^2} \frac{1}{1-z}, \tag{57}$$

$$\Delta_2^{(1)} = -\frac{1}{16\pi^2} \frac{\log(1-z)}{z}. \tag{58}$$

We will write down $G_{aba'b'}^{(S)}$ as a function of z explicitly with this choice of β . For this purpose the following formulas are useful [19]:

$$\nabla_a n_b = H \cot(\mu H) (g_{ab} - n_a n_b) = \frac{H(2z-1)}{2\sqrt{z(1-z)}} (g_{ab} - n_a n_b), \quad (59)$$

$$\nabla_a n_{b'} = -\frac{H}{\sin(\mu H)} (g_{ab'} + n_a n_{b'}) = -\frac{H}{2\sqrt{z(1-z)}} (g_{ab'} + n_a n_{b'}), \quad (60)$$

$$\nabla_a g_{bb'} = \frac{H[1 - \cos(\mu H)]}{\sin(\mu H)} (g_{ab} n_{b'} + g_{ab'} n_b) = H \sqrt{\frac{1-z}{z}} (g_{ab} n_{b'} + g_{ab'} n_b). \quad (61)$$

By using these formulae one finds [6]

$$\left(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) \Phi(z) = H^2 z(1-z) \frac{d^2 \Phi}{dz^2} \left(n_a n_b - \frac{1}{4} g_{ab} \right). \quad (62)$$

We also use

$$\begin{aligned} & \left(\nabla_{a'} \nabla_{b'} - \frac{1}{4} g_{a'b'} \square' \right) \left(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) \Phi(z) \\ &= H^4 \left\{ z(1-z) \frac{d^2}{dz^2} \left[z(1-z) \frac{d^2 \Phi}{dz^2} \right] T_{aba'b'}^{(1)} \right. \\ & \quad \left. + \frac{1}{4} \frac{d^2 \Phi}{dz^2} T_{aba'b'}^{(2)} + \frac{1-z}{2} \frac{d}{dz} \left[z \frac{d^2 \Phi}{dz^2} \right] T_{aba'b'}^{(3)} \right\}. \end{aligned} \quad (63)$$

The result is then given in the following form:

$$G_{aba'b'}^{(S)}(x, x')|_{\beta=2/3} = \frac{H^2}{16\pi^2} \sum_{k=1}^5 f^{(S,k)}(z) T_{aba'b'}^{(k)}, \quad (64)$$

where $T_{aba'b'}^{(4)}$ and $T_{aba'b'}^{(5)}$ are defined as

$$T_{aba'b'}^{(4)} = g_{ab} \left(n_{a'} n_{b'} - \frac{1}{4} g_{a'b'} \right) + g_{a'b'} \left(n_a n_b - \frac{1}{4} g_{ab} \right), \quad (65)$$

$$T_{aba'b'}^{(5)} = g_{ab} g_{a'b'}. \quad (66)$$

The functions $f^{(S,k)}(z)$ are

$$f^{(S,1)}(z) = \frac{4(\alpha-9)}{9} \left[\left(\frac{2}{z} - \frac{8}{z^2} + \frac{6}{z^3} \right) \log(1-z) - \frac{5}{z} + \frac{6}{z^2} \right], \quad (67)$$

$$f^{(S,2)}(z) = \left(\frac{\alpha}{18} - \frac{3}{2} \right) \frac{1}{1-z} + \frac{\alpha-9}{18} \left[\frac{2}{z^3} \log(1-z) + \frac{1}{z} + \frac{2}{z^2} \right], \quad (68)$$

$$f^{(S,3)}(z) = \left(\frac{\alpha}{9} - 3 \right) \frac{1}{1-z} + \frac{2(\alpha-9)}{9} \left[\left(\frac{2}{z^2} - \frac{2}{z^3} \right) \log(1-z) + \frac{1}{z} - \frac{2}{z^2} \right], \quad (69)$$

$$f^{(S,4)}(z) = \left(1 - \frac{\alpha}{9} \right) \left[\frac{1}{1-z} - \frac{2}{z} + \left(\frac{2}{z} - \frac{2}{z^2} \right) \log(1-z) \right], \quad (70)$$

$$f^{(S,5)}(z) = \left(\frac{\alpha}{36} - \frac{1}{12} \right) \frac{1}{1-z} + \left(\frac{\alpha}{18} - \frac{1}{2} \right) \frac{\log(1-z)}{z}. \quad (71)$$

The $G_{aba'b'}^{(S)}$ simplifies considerably if we let $\alpha = 9$.

5 Summary and Discussions

The full propagator can be obtained by adding the transverse-traceless, vector and scalar sectors as

$$G_{aba'b'}(x, x')|_{\beta=2/3} = \frac{H^2}{16\pi^2} \sum_{k=1}^5 [f^{(TV,k)}(z) + f^{(S,k)}(z)] T_{aba'b'}^{(k)} \quad (72)$$

with the definition $f^{(TV,4)}(z) = f^{(TV,5)}(z) = 0$.

Let us first comment on the relation between the results here and our previous work [15] which concentrated on the scalar sector. In [15] we did not adopt the Euclidean approach, and therefore the definition of the scalar sector was slightly different. Let us assume $1 - z \neq 0$ so that the two points are not on the light cone of one another. Then we have

$$\begin{aligned} (\square - m^2 H^2) \Delta_{m^2} &= 0, \\ (\square - m^2 H^2) \Delta_{m^2}^{(1)} &= -\Delta_{m^2}, \\ \square \Delta_0^- &= \frac{3H^4}{8\pi^2}. \end{aligned}$$

By using these formulas one can write the scalar sector as

$$G_{aba'b'}^{(S)}(x, x') = G_{aba'b'}^{(S1)}(x, x') + G_{aba'b'}^{(S2)}(x, x'), \quad (73)$$

where

$$\begin{aligned} G_{aba'b'}^{(S1)} &= \nabla_a \nabla_b \nabla_{a'} \nabla_{b'} \left(\frac{3-\alpha}{9H^4} \Delta_{3\beta} + \frac{4-(\alpha-3)\beta}{3H^2} \Delta_{3\beta}^{(1)} \right) \\ &\quad + (g_{ab} \nabla_{a'} \nabla_{b'} + g_{a'b'} \nabla_a \nabla_b) \frac{1}{3H^2} \Delta_{3\beta}, \end{aligned} \quad (74)$$

$$\begin{aligned} G_{aba'b'}^{(S2)} &= \frac{\alpha}{9H^4} \nabla_a \nabla_b \nabla_{a'} \nabla_{b'} \Delta_0^- \\ &\quad - \frac{1}{3H^4} \left(\nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) \left(\nabla_{a'} \nabla_{b'} - \frac{1}{4} g_{a'b'} \square' \right) \Delta_{-4}^-. \end{aligned} \quad (75)$$

The $G_{aba'b'}^{(S1)}$ is identical with the scalar sector of [15] for $\beta = 4/(\alpha - 3)$. The $G_{aba'b'}^{(S2)}$ clearly satisfies $g^{ab} G_{aba'b'}^{(S2)} = 0$ and can also be shown to satisfy $\nabla^a \nabla^b G_{aba'b'}^{(S2)} = 0$. Therefore, the non-scalar sector as defined in [15] is $G_{aba'b'}^{(TV)} + G_{aba'b'}^{(S2)}$. It is also clear that the terms that depend on the gauge parameters are of pure-gauge form, i.e. $\nabla_{(a} W_{b)a'b'}^{(1)} + \nabla_{(a'} W_{b')ab}^{(2)}$ for some $W_{ba'b'}^{(1)}$ and $W_{b'ab}^{(2)}$, where (\dots) denotes symmetrization. As expected, the two-point function of a gauge-invariant quantity does not depend on the gauge parameters because the pure-gauge part of the propagator will not contribute.

Finally, let us show that the two-point function of a gauge-invariant quantity obtained by applying differential operators on the field h_{ab} , which we call a local gauge-invariant quantity, is bounded as $z \rightarrow -\infty$ even though the propagator given by (72) is not bounded for any α . (One example of a local gauge-invariant quantity is the linearized Weyl tensor.) By choosing $\alpha = 0$ the linear term in $G_{aba'b'}$ is eliminated, and its growth for large $|z|$ is logarithmic. We will see that this growth is not reflected in the two-point function of a local gauge-invariant quantity.

First we note that the bi-vector

$$A_{aa'} = \phi_1(z)g_{aa'} + \phi_2(z)n_a n_{a'} \quad (76)$$

is divergence-free if

$$\phi_1(z) = -\frac{2}{3}z(1-z)f(z) + (2z-1) \int^z f(z)dz, \quad (77)$$

$$\phi_2(z) = -\frac{2}{3}z(1-z)f(z) + 2(z-1) \int^z f(z)dz \quad (78)$$

for some function $f(z)$ [6]. Next we find

$$\begin{aligned} & \nabla_a \nabla_{a'} A_{bb'} + \nabla_b \nabla_{a'} A_{ab'} + \nabla_a \nabla_{b'} A_{ba'} + \nabla_b \nabla_{b'} A_{aa'} \\ &= F^{(1)}(z)T_{aba'b'}^{(1)} + F^{(2)}(z)T_{aba'b'}^{(2)} + F^{(3)}(z)T_{aba'b'}^{(3)}, \end{aligned} \quad (79)$$

where

$$F^{(1)}(z) = -\frac{16}{3}z(1-z)f'(z), \quad (80)$$

$$F^{(2)}(z) = \frac{4}{3}(2z-1)f(z) - \frac{2}{3}z(1-z)f'(z), \quad (81)$$

$$F^{(3)}(z) = -\frac{2}{3}z^2(1-z)^2f''(z) + \frac{4}{3}(3z-2)z(1-z)f'(z) - \frac{8}{3}(1-z)^2f(z). \quad (82)$$

(The left-hand side of (79) is guaranteed to be traceless and, therefore, is a linear combination of $T_{aba'b'}^{(1)}$, $T_{aba'b'}^{(2)}$ and $T_{aba'b'}^{(3)}$, because $A_{aa'}$ is divergence-free.) By using (79) with $f(z) = 1/(1-z) + 1/[2(1-z)^2]$ we find that the bi-tensor

$$P_{aba'b'} \equiv \left[\frac{16}{3} - \frac{16}{3(1-z)^2} \right] T_{aba'b'}^{(1)} - 2T_{aba'b'}^{(2)} + \left[-2 - \frac{4}{3(1-z)} - \frac{2}{3(1-z)^2} \right] T_{aba'b'}^{(3)} \quad (83)$$

is of pure-gauge form. Notice that for large $|z|$ we have (for $\beta > 0$)

$$G_{aba'b'}|_{\alpha=0} \approx -\frac{H^2}{16\pi^2} \log(1-z) P_{aba'b'}, \quad (84)$$

where $P_{aba'b'}$ is bounded as $z \rightarrow -\infty$. Now, suppose that we want to find the two-point function of a local gauge-invariant quantity at $z = z_0$. We can use the propagator

$$G_{aba'b'}(x, x')|_{\alpha=0} + \frac{H^2}{16\pi^2} \log(1 - z_0) P_{aba'b'}(x, x')$$

with z_0 fixed for this calculation because $P_{aba'b'}$ is of pure-gauge form. This propagator and its derivatives at $z = z_0$ are bounded functions of z_0 as $|z_0|$ becomes large. This implies that the logarithmic growth of the propagator will not be reflected in the two-point function of a local gauge-invariant quantity, and that the latter is bounded as $z \rightarrow -\infty$.

Although this argument shows that two-point functions of local gauge-invariant quantities do not grow at large distances, it does not show whether or not they decrease, or if so, how rapidly. One needs to compute such quantities explicitly to find their detailed large-distance behaviour. One of the authors (SSK) is currently investigating the two-point function of the linearized Weyl tensor. The result of this calculation will be reported elsewhere [20].

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